2^{nd}-Order Homogeneous ODE with Constant Coefficients Differential Equations X. Du

- Take the form: $ay''+by'+cy=0$
- How to solve:
  - Guess: $y=e^{rt}$ — Substitute
  - $ar^2e^{rt}+bre^{rt}+ce^{rt}=0$ — e^{rt} is never 0, so cancel
  - $ar^2+br+c=0$ — This is the **characteristic equation** for *r*.
  - $r=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ — Use quadratic formula to solve for *r*.
- Cases:
  - **Overdamped**: two distinct, real roots (i.e. $b^2-4ac>0$)
    - Let r_1 and r_2 solve the characteristic equation. $r_1\neq r_2$
    - $\therefore y_1=e^{r_1t}$ and $y_2=e^{r_2t}$ are solutions to $ay''+by'+cy=0$
    - General solution: $y=c_1e^{r_1t}+c_2e^{r_2t}$ (apply superposition)
  - **Critically damped**: one repeated real root (i.e. $b^2-4ac=0$)
    - General solution: $y=c_1e^{rt}+c_2te^{rt}$
    - This is most easily proved with the Laplace transform
  - **Underdamped**: two complex roots (i.e. $b^2-4ac<0$)
    - Let the roots be $r=a\pm bi$
    - General solution: $y=e^{at}(c_1\cos bt+c_2\sin bt)$
    - Theorem: If a complex solution *y* solves a linear homogeneous differential equation, then $\operatorname{Re}[y]$ and $\operatorname{Im}[y]$ are also solutions. Proof with substitution and superposition.
    - Proof of general solution:
      - $y=e^{(a\pm bi)t}=e^{at}e^{i(\pm bt)}$
      - $y=e^{at}cis(\pm bt)$. Use Euler's formula.
      - $y_1=e^{at}\cos(bt)$ and $y_2=\pm e^{at}\sin(bt)$. Apply theorem.
      - $y=e^{at}(c_1\cos bt+c_2\sin bt)$. Apply superposition.
    - **Free**: $b=0$
      - Let the roots be $r=\pm bi$
      - General solution $y=c_1\cos bt+c_2\sin bt$
- Applications
  - Unforced mass-spring system with damping $mx''+cx'+kx=0$
  - RLC circuit: $Lq''+Rq'+\dfrac{q}{C}=0$; LC circuit: $Lq''+\dfrac{q}{C}=0$
  - Simple harmonic motion.
    - Pendulums (approximated model using the small angle approximation $\theta''+\dfrac{g}{L}\theta=0$). $\theta''+\dfrac{g}{L}\sin\theta=0$ is the more accurate model.
    - Unforced mass-spring system without damping $mx''+kx=0$